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Field of a charged particle in Brans–Dicke theory of gravitation

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Abstract. Field equations in the Brans–Dicke scalar–tensor theory of gravitation are obtained for a static charged point mass with the aid of a spherically symmetric, conformally flat metric. A closed-form exact solution of the field equations is presented, and may be considered as describing the field due to a charged mass point at the origin surrounded by a scalar–tensor field in a conformally flat space.

1. Introduction

This work is a continuation of Reddy (1979), of the Brans–Dicke (BD) (1961) theory of gravitation, in which we have obtained spherically symmetric, static, conformally flat solutions of the BD vacuum field equations. Here we have considered the energy–momentum tensor due to a source-free electromagnetic field, and have obtained an exact solution of the BD–Maxwell field equations for a static charged point mass with the aid of a conformally flat metric. We seek exact solutions to the BD–Maxwell field equations with this specialised metric form; partly because exact solutions of BD gravitational field equations are scarce, but also because conformally flat metrics have cosmological interest (Synge 1960). Thus any such exact solutions may be useful, even though somewhat unrealistic. It may be mentioned, in this context, that Penney (1969) has solved the field equations for coupled gravitational and zero-mass scalar fields in the presence of a point charge in the spherically symmetric static case.

In § 2, we have set up the BD–Maxwell field equations in a suitable form with the aid of a spherically symmetric, static, conformally flat metric. In § 3, an exact solution to the BD–Maxwell field equations is obtained and studied. The last section contains some concluding remarks.

2. Field equations

The BD field equations for a source-free electromagnetic field are

$$R_{ij}-\tfrac{1}{2}g_{ij}R=8\pi\phi^{-1}T_{ij}+\omega\phi^{-2}(\phi_{,i}\phi_{,j}-\tfrac{1}{2}g_{ij}\phi^{,k}\phi_{,k})+\phi^{-1}\phi_{;ij}, \qquad (1)$$

$$\phi^{;k}_{;k}=0 \qquad (\omega\neq-\tfrac{3}{2}), \qquad (2)$$

$$F^{ij}_{;j}=0 \qquad (3)$$

and

$$F_{ij,k}+F_{jk,i}+F_{ki,j}=0 \tag{4}$$

with

$$T_{ij}=F_{il}F^{l}_{j}-\tfrac{1}{4}g_{ij}F_{lm}F^{lm} \tag{5}$$

where R_{ij} is the Ricci tensor, T_{ij} is the energy–momentum tensor due to a source-free electromagnetic field, F_{ij} is the electromagnetic field tensor, ϕ is the scalar field, ω is the coupling constant and a comma and semicolon denote partial and covariant derivatives respectively.

In the following section, we solve the field equations (1)–(4) for a static, spherically symmetric, conformally flat metric

$$ds^2=e^{\alpha}(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\Phi^2-dt^2) \tag{6}$$

where α is a function of r alone. The spherical symmetry assumed implies that the BD scalar field ϕ shares the same symmetry.

From (3), (5) and (6) the non-vanishing components of the energy–momentum tensor, for the electrostatic case, are found to be

$$T^1_1=-T^2_2=-T^3_3=T^4_4=-\tfrac{1}{2}e^{-2\alpha}q^2/r^4 \tag{7}$$

where q is a constant which is identified with the electric charge.

Taking ϕ to be a function of r only, and using (6) and (7) in (1) and (2), the BD field equations for a charged point mass can be written, explicitly, as

$$\tfrac{3}{4}\alpha'^2+\frac{2\alpha'}{r}=-4\pi\phi^{-1}\frac{q^2}{r^4}e^{-\alpha}+\frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2+\frac{\phi''}{\phi}-\frac{\alpha'\phi'}{2\phi}, \tag{8}$$

$$\alpha''+\frac{\alpha'^2}{4}+\frac{\alpha'}{r}=4\pi\phi^{-1}\frac{q^2}{r^4}e^{-\alpha}-\frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2+\frac{\phi'}{\phi}\left(\frac{\alpha'}{2}+\frac{1}{r}\right), \tag{9}$$

$$\alpha''+\frac{\alpha'^2}{4}+\frac{2\alpha'}{r}=-4\pi\phi^{-1}\frac{q^2}{r^4}e^{-\alpha}-\frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2+\frac{\alpha'\phi'}{2\phi}, \tag{10}$$

$$\phi''+\phi'(\alpha'+2/r)=0. \tag{11}$$

Here a superscript prime indicates differentiation with respect to r.

3. Solutions of the field equations

It can be easily verified, from the field equations, that when the scalar field is a constant and the charge $q=0$ we obtain the empty flat space–time in Einstein's theory.

When ϕ is a function of r only and when $q=0$ the field equations yield the static, spherically symmetric, conformally flat vacuum solution, obtained by Reddy (1979) for BD theory, which is given by

$$ds^2=e^{\alpha}(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\Phi^2-dt^2) \tag{12}$$

with $\alpha=C/r$ and $\phi=\phi_0\exp(-C/r)$, where C and ϕ_0 are constants of integration.

It can also be seen, from the field equations, that when ϕ is constant and the charge q is non-zero we arrive at the solution given by

$$ds^2=e^{\alpha}(dr^2+r^2\,d\theta^2+r^2\sin^2\theta\,d\Phi^2-dt^2)$$

with

$$e^{\alpha} = 1 + 4\pi G_0 q^2/r^2, \qquad \phi = \text{constant} = \phi_0, \tag{13}$$

where $G_0 = \phi_0^{-1}$ is the usual gravitational constant. The solution (13) may be looked upon as the Reissner–Nordström type solution in Einstein's theory in a conformally flat space–time.

When the BD scalar field ϕ is a function of r only and when the charge q is not equal to zero, we solve the field equations (8)–(11). Since the field equations are four and the unknowns are two only, the question of overdeterminacy is settled by satisfaction of all the field equations by actual substitution of the solution derived.

It is a simple matter to see that the field equations (9)–(11) reduce to

$$r^2 e^{\alpha}\phi' = C_1, \qquad C_1 \neq 0,$$

and

$$r^2 e^{\alpha}\phi = C_2 r^2 + C_3, \qquad C_2 \neq 0, \tag{14}$$

which in turn gives us

$$\phi'/\phi = C_1/(C_2 r^2 + C_3) \tag{15}$$

where C_1, C_2 are constants of integration and C_3 is set equal to $4\pi q^2$.

From (14) and (15) it is easy to see that

$$e^{\alpha} = \left(C_2 + \frac{C_3}{r^2}\right)\phi_0^{-1} \exp\left(\frac{-C_1}{(-C_2C_3)^{1/2}} \tanh^{-1} \frac{(-C_2C_3)^{1/2}}{C_3} r\right)$$

and

$$\phi = \phi_0 \exp\left(\frac{C_1}{(-C_2C_3)^{1/2}} \tanh^{-1} \frac{(-C_2C_3)^{1/2}}{C_3} r\right) \tag{16}$$

where the constants ϕ_0, C_2 and C_3 are strictly non-zero. It is seen that the solution (16) satisfies each of the field equations (8)–(11) provided the constants C_1, C_2, C_3 are related by

$$(3 + 2\omega)C_1^2 + 12 C_2 C_3 = 0. \tag{17}$$

It can be seen that equation (17), together with $\omega > 0$, $C_1^2 > 0$ and $C_3 > 0$, implies $C_2 < 0$. Therefore $(-C_2C_3)^{1/2}$ in (16) becomes real. Thus equations (6) and (16) along with (17) constitute an exact, static, spherically symmetric, conformally flat solution of the BD–Maxwell field equations. Physically, this solution may be looked upon as describing the field of a charged particle at the origin surrounded by the scalar–tensor field in a conformally flat space–time.

It can be seen that as $r \to \infty$, the solution (16) goes over to empty flat space–time of Einstein's theory with $\phi = \text{constant}$. It is interesting to see that the vacuum solution of Reddy (1979), given by (12) in the case of BD theory, can only be obtained from (15) and (14) by setting $C_3 = 4\pi q^2 = 0$ (i.e. when the electric charge is put off), since the exact, explicit solution (16) along with (17) is valid only for non-zero C_2 and C_3.

4. Conclusion

In order to understand more fully the equations of the BD theory of gravitation, it is useful to have a knowledge of some exact solutions of these equations. Here we have obtained a closed-form, exact, static, spherically symmetric, conformally flat solution of the coupled BD–Maxwell field equations corresponding to a charged mass point at the origin surrounded by the scalar–tensor field. The solution (16) will play a physically significant role in studies of the interaction of electromagnetic and scalar–tensor gravitational fields in a conformally flat space–time. Also, such solutions will help in the study of the BD gravitational theory and the relationship between this theory and that of Einstein.

In conclusion, we wish to mention that conformally flat metrics have cosmological interest (Synge 1960), and we hope that some physical insight can be gained from the solution obtained in this paper.

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